

Sharp integral inequalities for the dyadic maximal operator and applications

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Abstract

We prove a sharp integral inequality for the dyadic maximal function of $\phi \in L^p$. This inequality connects certain quantities related to integrals of ϕ and the dyadic maximal function of ϕ , under the hypothesis that the variables $\int_X \phi d\mu = f$, $\int_X \phi^q d\mu = A$, $1 < q < p$, are given, where $0 < f^q \leq A$. Additionally, it contains a parameter $\beta > 0$ which when it attains a certain value depending only on f, A, q , the inequality becomes sharp. Using this inequality we give an alternative proof of the evaluation of the Bellman function related to the dyadic maximal operator of two integral variables.

1 Introduction

It is well known that the dyadic maximal operator on \mathbb{R}^n is a useful tool in analysis and is defined by

$$\mathcal{M}_d\phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(y)| dy : x \in Q, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\}, \quad (1.1)$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$, where the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, for $N = 0, 1, 2, \dots$. It is also well known that it satisfies the following weak type (1,1) inequality

$$\left| \{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) > \lambda\} \right| \leq \frac{1}{\lambda} \int_{\{\mathcal{M}_d\phi > \lambda\}} |\phi(y)| dy, \quad (1.2)$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$, and which is easily proved to be best possible. Further refinements of (1.2) can be seen in [9] and [10].

Then by using (1.2) and the well known Doob's method it is not difficult to prove that the following L^p inequality is also true

$$\|\mathcal{M}_d\phi\|_p \leq \frac{p}{p-1} \|\phi\|_p, \quad (1.3)$$

⁰Keywords: Bellman, dyadic maximal function, integral inequality

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⁰MSC Number: 42B25

for every $p > 1$ and $\phi \in L^p(\mathbb{R}^n)$. Inequality (1.3) turns out to be best possible and its sharpness is proved in [16] (for general martingales see [1] and [2]).

One way to study inequalities satisfied by maximal operators is by using the so called Bellman function technique. For example, in order to refine (1.3) we can insert the L^1 -norm of ϕ as an independent variable in (1.3), and try to find the best possible upper bound of $\|\mathcal{M}_d\phi\|_p$, when both the L^1 and L^p norms of ϕ are given, by evaluating the (Bellman) function of two variables

$$B^{(p)}(f, F) = \sup \left\{ \frac{1}{|Q|} \int_Q (\mathcal{M}_d\phi)^p : \phi \geq 0, \frac{1}{|Q|} \int_Q \phi = f, \frac{1}{|Q|} \int_Q \phi^p = F \right\}, \quad (1.4)$$

where Q is a fixed dyadic cube and f, F are such that $0 < f^p \leq F$.

The approach of studying maximal operators by the introduction of the corresponding Bellman function was first seen in the work of Nazarov and Treil, [5], where the authors defined the function

$$B_p(f, F, L) = \sup \left\{ \frac{1}{|Q|} \int_Q (\mathcal{M}_d\phi)^p : \frac{1}{|Q|} \int_Q \phi = f, \frac{1}{|Q|} \int_Q \phi^p = F, \sup_{R: Q \subseteq R} \frac{1}{|R|} \int_R \phi = L \right\}, \quad (1.5)$$

with $p > 1$ (as an example they examine the case $p = 2$), Q is as above and ϕ is non-negative in $L^p(Q)$, R runs over all dyadic cubes containing Q and the variables F, f, L satisfy $0 \leq f \leq L$, $f^p \leq F$. Exploiting a certain "pseudoconcavity" inequality it satisfies, they construct the function $4F - 4fL + 2L^2$ which has the same properties as (1.5) and provides a good L^p bound for the operator \mathcal{M}_d (see [5] for details).

Both of the above Bellman functions were explicitly computed for the first time by Melas in [3]. In fact this was done in the much more general setting of a non-atomic probability space (X, μ) equipped with a tree structure \mathcal{T} , which is similar to the structure of the dyadic subcubes of $[0, 1]^n$ (see the definition in Section 2). Then the associated maximal operator is defined as

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\}, \quad (1.6)$$

for every $\phi \in L^1(X, \mu)$. Moreover (1.2) and (1.3) still hold in this setting and remain sharp. Now if we wish to refine (1.3) for the general case of a tree \mathcal{T} , we should introduce the Bellman function of two variables related to the above maximal operator, which is given by

$$B_{\mathcal{T}}^{(p)}(f, F) = \sup \left\{ \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^p d\mu = F \right\}, \quad (1.7)$$

where $0 < f^p \leq F$. This function of course generalizes (1.4). In [3] it is proved that

$$B_{\mathcal{T}}^{(p)}(f, F) = F \omega_p \left(\frac{f^p}{F} \right)^p, \quad (1.8)$$

where $\omega_p : [0, 1] \rightarrow [1, \frac{p}{p-1}]$, is defined by $\omega_p(z) = H_p^{-1}(z)$, and $H_p(z)$ is given by $H_p(z) = -(p-1)z^p + pz^{p-1}$. As a consequence $B_{\mathcal{T}}^{(p)}(f, F)$ does not depend on the tree \mathcal{T} . The technique for the evaluation of (1.7), that is used in [3], is based on an effective linearization of the dyadic maximal operator that holds on an adequate class of functions called \mathcal{T} -good (see the definition in Section 2), which is enough to describe the problem as is settled in (1.7). Using this result on suitable subsets of X and several calculus arguments, the author also managed to precisely evaluate the corresponding to (1.5) Bellman function in this context,

$$B_{\mathcal{T}}^p(f, F, L) = \sup \left\{ \int_X (\max(\mathcal{M}_{\mathcal{T}}\phi, L))^p d\mu : \phi \geq 0, \phi \in L^p(X, \mu), \right. \\ \left. \int_X \phi d\mu = f, \int_X \phi^p d\mu = F, \right\}. \quad (1.9)$$

Now (1.7) and (1.9) were computed in [8] in a different way that avoids the calculus arguments involved in [3]. A crucial intermediate result the authors obtain there, in this direction, is the following.

Theorem A. *Let $\phi \in L^p(X, \mu)$ be non-negative, with $\int_X \phi d\mu = f$. Then the following inequality is true*

$$\int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu \leq -\frac{1}{p-1}f^p + \frac{p}{p-1} \int_X \phi (\mathcal{M}_{\mathcal{T}}\phi)^{p-1} d\mu. \quad (1.10)$$

The motivation for our work here comes from our wish to refine (1.7) even further by also considering the q -norm, $1 < q < p$, of the function ϕ as fixed and to compute the corresponding Bellman function. In particular, our goal was the evaluation of

$$B_{\mathcal{T}}^{p,q}(f, A, F) = \sup \left\{ \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu : \phi \geq 0, \phi \in L^p(X, \mu), \right. \\ \left. \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \int_X \phi^p d\mu = F, \right\}, \quad (1.11)$$

where $1 < q < p$, and for f, A, F we have $f^q < A < F^{\frac{q}{p}}$. The new integral variable makes the problem considerably more difficult.

In Sections 3 and 4 we prove our main result, stated in Theorem 1 below. It is an inequality which we believe that it can be the corresponding to Theorem A intermediate step in the present context, towards the evaluation of (1.11). Then using this result and entangling a result from [3] we prove Corollary 1 below which directly strengthens and generalizes Theorem A. This will be carried out in Section 5. Finally, also in Section 5, we exploit these results to evaluate (1.7) in a new way.

So our main result is the following.

Theorem 1. *Let $q \in (1, p)$, $f > 0$ and $\phi \in L^p(X, \mu)$ non-negative, with $\int_X \phi d\mu = f$. Then the inequality*

$$\begin{aligned} \int_X (\mathcal{M}_T \phi)^p d\mu &\leq \frac{p(\beta+1)^q}{G(p, q, \beta)} \int_X (\mathcal{M}_T \phi)^{p-q} \phi^q d\mu + \frac{(p-q)(\beta+1)}{G(p, q, \beta)} f^p \\ &\quad + \frac{p(q-1)\beta}{G(p, q, \beta)} f^{p-q} \int_X (\mathcal{M}_T \phi)^q d\mu - \frac{p(\beta+1)^q}{G(p, q, \beta)} f^{p-q} \int_X \phi^q d\mu, \end{aligned} \quad (1.12)$$

where $G(p, q, \beta) = p(q-1)\beta + (p-q)(\beta+1)$, is sharp for every $\beta > 0$. If we also assume that $\int_X \phi^q d\mu = A$, $f^q < A$, then (1.12) is best possible for $\beta = \omega_q(\frac{f^q}{A}) - 1$, where ω_q is defined as above, with q in place of p .

Theorem 1, together with results from [3], will allow us to prove the following generalization of Theorem A, which in turn will lead to the evaluation of (1.7) in a new way.

Corollary 1. *Let $q \in (1, p)$, $f > 0$, and $\phi \in L^p(X, \mu)$ non-negative, with $\int_X \phi d\mu = f$. Then the inequality*

$$\int_X (\mathcal{M}_T \phi)^p d\mu \leq -\frac{q(\beta+1)}{G(p, q, \beta)} f^p + \frac{p(\beta+1)^q}{G(p, q, \beta)} \int_X (\mathcal{M}_T \phi)^{p-q} \phi^q d\mu, \quad (1.13)$$

is sharp for every $\beta > 0$, where $G(p, q, \beta)$ as above. If we also assume that $\int_X \phi^q d\mu = A$, $f^q < A$, then (1.13) is best possible for $\beta = \omega_q(\frac{f^q}{A}) - 1$,

We remark here that there are several problems in Harmonic Analysis where Bellman functions arise. Such problems (including the dyadic Carleson imbedding theorem and weighted inequalities) are described in [7] (see also [5], [6]) and also connections to Stochastic Optimal Control are provided, from which it follows that the corresponding Bellman functions satisfy certain nonlinear second-order PDEs. The exact evaluation of a Bellman function is a difficult task which is connected with the deeper structure of the corresponding Harmonic Analysis problem. Until now several Bellman functions have been computed (see [1], [3], [5], [12], [13], [14]). The exact computation of (1.7) has also been given in [11] by L. Slavin, A. Stokolos and V. Vasyunin, which linked the computation of it to solving certain PDEs of the Monge-Ampère type, and in this way they obtained an alternative proof of the results in [3] for the Bellman function related to the dyadic maximal operator. Also in [15], using the Monge-Ampère equation approach, a more general Bellman function than the one related to the Carleson imbedding theorem has been precisely evaluated thus generalizing the corresponding result in [3]. It would be an interesting problem to discover if the Bellman function of three variables defined in (1.11) can be computed using such PDE-based methods.

2 Preliminaries

In this section we present (without proofs) the background we need from [3], that will be used in all that follows.

Let (X, μ) be a non-atomic probability space. Two measurable subsets A, B of X will be called almost disjoint if $\mu(A \cap B) = 0$.

Definition 2.1. A set \mathcal{T} of measurable subsets of X will be called a tree if the following conditions are satisfied:

- (i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.
- (ii) For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $\mathcal{C}(I) \subseteq \mathcal{T}$ containing at least two elements such that:
 - (a) the elements of $\mathcal{C}(I)$ are pairwise almost disjoint subsets of I ,
 - (b) $I = \bigcup \mathcal{C}(I)$.
- (iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_m$ where $\mathcal{T}_0 = \{X\}$ and $\mathcal{T}_{m+1} = \bigcup_{I \in \mathcal{T}_m} \mathcal{C}(I)$.
- (iv) We have $\lim_{m \rightarrow \infty} \sup_{I \in \mathcal{T}_m} \mu(I) = 0$.
- (v) \mathcal{T} differentiates $L^1(X, \mu)$

This last condition means exactly that the Lebesgue differentiation theorem holds in the space $L^1(X, \mu)$, with respect to the tree \mathcal{T} .

Now we define for any tree \mathcal{T} its exceptional set $E = E(\mathcal{T})$ as follows:

$$E(\mathcal{T}) = \bigcup_{I \in \mathcal{T}} \bigcup_{\substack{J_1, J_2 \in \mathcal{C}(I) \\ J_1 \neq J_2}} (J_1 \cap J_2). \quad (2.1)$$

It is easy to see that $E(\mathcal{T})$ has measure 0.

By induction it can be seen that each family \mathcal{T}_m consists of pairwise almost disjoint sets whose union is X . Moreover if $x \in X \setminus E(\mathcal{T})$ then for each m there exists exactly one $I_m(x)$ in \mathcal{T}_m containing x . For every $m > 0$ there is a $J \in \mathcal{T}_{m-1}$ such that $I_m(x) \in \mathcal{C}(J)$. Then, since $x \in J$, we must have $J = I_{m-1}(x)$, because x does not belong to $E(\mathcal{T})$. Hence the set $\mathcal{A} = \{I \in \mathcal{T} : x \in I\}$ forms a chain $I_0(x) = X \supsetneq I_1(x) \supsetneq \dots$ with $I_m \in \mathcal{C}(I_{m-1}(x))$ for every $m > 0$. From this remark it follows that if $I, J \in \mathcal{T}$ and $I \cap J \cap (X \setminus E(\mathcal{T}))$ is nonempty, then $I \subseteq J$ or $J \subseteq I$. In particular for any $I, J \in \mathcal{T}$, either $\mu(I \cap J) = 0$ or one of them is contained in the other.

Given any tree \mathcal{T} we remind that the maximal operator associated to it is defined as follows:

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\left\{\frac{1}{\mu(I)} \int_I |\phi| \, d\mu : x \in I \in \mathcal{T}\right\}, \quad (2.2)$$

for every $\phi \in L^1(X, \mu)$.

Next we describe the linearization procedure for the operator $\mathcal{M}_{\mathcal{T}}$. Let $\phi \in L^1(X, \mu)$ be a nonnegative function and for any $I \in \mathcal{T}$ let

$$\text{Av}_I(\phi) = \frac{1}{\mu(I)} \int_I \phi \, d\mu. \quad (2.3)$$

We will say that ϕ is \mathcal{T} -good if the set

$$\Lambda_\phi = \{x \in X \setminus E(\mathcal{T}) : \mathcal{M}_\mathcal{T}\phi(x) > \text{Av}_I(\phi) \text{ for all } I \in \mathcal{T} \text{ such that } x \in I\} \quad (2.4)$$

has μ -measure zero.

For any such function and every $x \in X \setminus (E(\mathcal{T}) \cup \Lambda_\phi)$ (i.e. for μ -almost every x in X) we define $I_\phi(x)$ to be the largest element in the nonempty set $\{I \in \mathcal{T} : x \in I \text{ and } \mathcal{M}_\mathcal{T}\phi(x) = \text{Av}_I(\phi)\}$.

Also given any $I \in \mathcal{T}$ let

$$A(\phi, I) = \{x \in X \setminus (E(\mathcal{T}) \cup \Lambda_\phi) : I_\phi(x) = I\} \subseteq I \quad (2.5)$$

and

$$\mathcal{S}_\phi = \{I \in \mathcal{T} : \mu(A(\phi, I)) > 0\} \cup \{X\}. \quad (2.6)$$

It is clear that

$$\mathcal{M}_\mathcal{T}\phi = \sum_{I \in \mathcal{S}_\phi} \text{Av}_I(\phi) \chi_{A(\phi, I)}, \text{ almost everywhere,} \quad (2.7)$$

where χ_B denotes the characteristic function of $B \subset X$. Now we define the correspondence $I \rightarrow I^*$ with respect to \mathcal{S}_ϕ for $I \neq X$ in the following manner: I^* is the smallest element of $\{J \in \mathcal{S}_\phi : I \subsetneq J\}$.

It is clear that the sets $A_I = A(\phi, I)$, $I \in \mathcal{S}_\phi$, are pairwise disjoint and since $\mu(\cup_{J \in \mathcal{S}_\phi} A_J) = 0$ their union has full measure.

In the following Lemma we present several important properties of the sets defined above. At this point we define two measurable sets A and B to be almost equal if $\mu(A \setminus B) = \mu(B \setminus A) = 0$ and in this case we write $A \approx B$

Lemma 2.1. (i) If $I, J \in \mathcal{S}_\phi$ then either $A_J \cap I = \emptyset$ or $J \subseteq I$.

(ii) If $I \in \mathcal{S}_\phi$ then there exists $J \in \mathcal{C}(I)$ such that $J \notin \mathcal{S}_\phi$

(iii) For every $I \in \mathcal{S}_\phi$ we have $I \approx \bigcup_{J \in \mathcal{S}_\phi : J \subseteq I} A_J$.

(iv) For every $I \in \mathcal{S}_\phi$ we have $A_I \approx I \setminus \bigcup_{J \in \mathcal{S}_\phi : J^* = I} A_J$ and so

$$\mu(A_I) = \mu(I) - \sum_{J \in \mathcal{S}_\phi : J^* = I} \mu(A_J). \quad (2.8)$$

From the above we get

$$\text{Av}_I(\phi) = \frac{1}{\mu(I)} \sum_{J \in \mathcal{S}_\phi : J \subseteq I} \int_{A_J} \phi \, d\mu. \quad (2.9)$$

Now we fix $q > 1$. Following [3] we set

$$x_I = a_I^{-1 + \frac{1}{q}} \int_{A_I} \phi \, d\mu \quad (2.10)$$

for every $I \in \mathcal{S}_\phi$ where $a_I = \mu(A_I)$ (in case where $\mu(A_X) = 0$ we set $x_X = 0$) and from Hölder's inequality and Lemma 2.1 we get

$$\mathcal{M}_\mathcal{T}\phi = \sum_{I \in \mathcal{S}_\phi} \left(\frac{1}{\mu(I)} \sum_{J \in \mathcal{S}_\phi: J \subseteq I} a_J^{1/\dot{q}} x_J \right) \chi_{A_I} \quad (2.11)$$

μ -almost everywhere, where $\dot{q} = \frac{q}{q-1}$ is the dual exponent of q , and also

$$\int_X \phi^q d\mu = \sum_{I \in \mathcal{S}_\phi} \int_{A_I} \phi^q d\mu \geq \sum_{I \in \mathcal{S}_\phi} x_I^q. \quad (2.12)$$

So we have

$$\int_X (\mathcal{M}_\mathcal{T}\phi)^q d\mu = \sum_{I \in \mathcal{S}_\phi} \left(\frac{1}{\mu(I)} \sum_{J \in \mathcal{S}_\phi: J \subseteq I} a_J^{1/\dot{q}} x_J \right)^q a_I = \sum_{I \in \mathcal{S}_\phi} a_I y_I^q \quad (2.13)$$

where

$$y_I = \text{Av}_I(\phi) = \frac{1}{\mu(I)} \sum_{J \in \mathcal{S}_\phi: J \subseteq I} a_J^{1/\dot{q}} x_J. \quad (2.14)$$

3 Proof of (1.12)

We shall first prove (1.12) for the class of \mathcal{T} -good functions. Let $\phi : (X, \mu) \rightarrow \mathbb{R}^+$ be \mathcal{T} -good and such that $\int_X \phi d\mu = f$ and $\int_X \phi^q d\mu = A$. We use the linearization technique mentioned in Section 2. From (2.11) and (2.14), if we set

$$F' = \int_X (\mathcal{M}_\mathcal{T}\phi)^{p-q} \phi^q d\mu,$$

we get

$$\begin{aligned} F' &= \int_X \sum_{I \in \mathcal{S}} y_I^{p-q} \chi_{A_I} \phi^q d\mu = \sum_{I \in \mathcal{S}} y_I^{p-q} \int_{A_I} \phi^q d\mu \\ &= \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} y_I^{p-q} \int_{A_I} \phi^q d\mu + y_X^{p-q} A - y_X^{p-q} \sum_{\substack{I \in \mathcal{S} \\ I^* = X}} \int_I \phi^q d\mu, \end{aligned} \quad (3.1)$$

where for the last equality in (3.1) we used Lemma ??(iv). Lemma ??(iii) and the definition of the correspondence $I \rightarrow I^*$ imply

$$\sum_{\substack{I \in \mathcal{S} \\ I^* = X}} \int_I \phi^q d\mu = \sum_{\substack{I \in \mathcal{S} \\ I^* = X}} \sum_{\substack{J \in \mathcal{S} \\ J \subseteq I}} \int_{A_J} \phi^q d\mu = \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} \int_{A_I} \phi^q d\mu. \quad (3.2)$$

Moreover, it is easy to see that

$$x_I^q = a_I^{q-1} (y_I \mu(I) - \sum_{\substack{J \in \mathcal{S} \\ J^* = I}} y_J \mu(J))^q$$

and

$$\int_{A_I} \phi^q d\mu \geq x_I^q.$$

So using Hölder's inequality in the form

$$\frac{(\lambda_1 + \lambda_2 + \dots + \lambda_m)^q}{(\sigma_1 + \sigma_2 + \dots + \sigma_m)^{q-1}} \leq \frac{\lambda_1^q}{\sigma_1^{q-1}} + \frac{\lambda_2^q}{\sigma_2^{q-1}} + \dots + \frac{\lambda_m^q}{\sigma_m^{q-1}}, \quad (3.3)$$

which holds for every $\lambda_i \geq 0$, $\sigma_i > 0$, since $q > 1$, Lemma ??(iv) and the properties of the correspondence $I \rightarrow I^*$, (3.1) becomes

$$\begin{aligned} F' &= \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} y_I^{p-q} \int_{A_I} \phi^q d\mu + y_X^{p-q} A - y_X^{p-q} \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} \int_{A_I} \phi^q d\mu \\ &= \sum_{I \in \mathcal{S}} (y_I^{p-q} - y_X^{p-q}) \int_{A_I} \phi^q d\mu + y_X^{p-q} A \\ &\geq \sum_{I \in \mathcal{S}} (y_I^{p-q} - y_X^{p-q}) x_I^q + y_X^{p-q} A \\ &= \sum_{I \in \mathcal{S}} (y_I^{p-q} - y_X^{p-q}) \frac{(y_I \mu(I) - \sum_{J^*=I} y_J \mu(J))^q}{(\mu(I) - \sum_{J^*=I} \mu(J))^{q-1}} + y_X^{p-q} A \\ &\geq \sum_{I \in \mathcal{S}} (y_I^{p-q} - y_X^{p-q}) \left(\frac{(y_I \mu(I))^q}{(\tau_I \mu(I))^{q-1}} - \sum_{\substack{J \in \mathcal{S} \\ J^*=I}} \frac{(y_J \mu(J))^q}{((\beta+1)\mu(J))^{q-1}} \right) + y_X^{p-q} A \\ &= K - \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} (y_{I^*}^{p-q} - y_X^{p-q}) \frac{(y_I \mu(I))^q}{((\beta+1)\mu(I))^{q-1}} \\ &= K - \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} y_{I^*}^{p-q} y_I^q \frac{\mu(I)^q}{((\beta+1)\mu(I))^{q-1}} + y_X^{p-q} \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} \frac{(y_I \mu(I))^q}{((\beta+1)\mu(I))^{q-1}} \quad (3.4) \end{aligned}$$

provided that the $\tau_I > 0$ satisfy $\tau_I \mu(I) - (\beta+1) \sum_{J^*=I} \mu(J) = \mu(I) - \sum_{J^*=I} \mu(J)$, which in turn gives

$$\tau_I = \beta + 1 - \beta \rho_I, \quad (3.5)$$

with $\rho_I = \frac{a_I}{\mu(I)}$, and

$$K = \sum_{I \in \mathcal{S}} (y_I^{p-q} - y_X^{p-q}) \frac{(y_I \mu(I))^q}{(\tau_I \mu(I))^{q-1}} + y_X^{p-q} A, \quad (3.6)$$

We now use the following elementary inequality,

$$px^q \cdot y^{p-q} \leq qx^p + (p-q)y^p,$$

which holds since $1 < q < p$, for any $x, y > 0$, to get

$$F' \geq K - \frac{p-q}{p} \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} y_{I^*}^p \frac{\mu(I)}{(\beta+1)^{q-1}} - \frac{q}{p} \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} y_I^p \frac{\mu(I)}{(\beta+1)^{q-1}} + y_X^{p-q} \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} \frac{y_I^q \mu(I)}{(\beta+1)^{q-1}}. \quad (3.7)$$

From Lemma 2.1 (iv),

$$\begin{aligned} \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} \mu(I) y_{I^*}^p &= \sum_{I \in \mathcal{S}} \sum_{\substack{J \in \mathcal{S} \\ J^* = I}} \mu(J) y_I^p = \sum_{I \in \mathcal{S}} (\mu(I) - a_I) y_I^p \\ &= y_X^p + \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} \mu(I) y_I^p - \sum_{I \in \mathcal{S}} a_I y_I^p. \end{aligned} \quad (3.8)$$

So

$$\begin{aligned} F' &\geq K - \sum_{\substack{I \in \mathcal{S} \\ I \neq X}} (y_I^{p-q} - y_X^{p-q}) \frac{y_I^q \mu(I)}{(\beta+1)^{q-1}} - \frac{(p-q) y_X^p}{p(\beta+1)^{q-1}} + \frac{p-q}{p} \sum_{I \in \mathcal{S}} \frac{a_I y_I^p}{(\beta+1)^{q-1}} \\ &= \sum_{I \in \mathcal{S}} (y_I^{p-q} - y_X^{p-q}) \frac{1}{\rho_I} \left(\frac{1}{(\beta+1-\beta\rho_I)^{q-1}} - \frac{1}{(\beta+1)^{q-1}} \right) a_I y_I^q \\ &\quad - \frac{p-q}{p} \frac{y_X^p}{(\beta+1)^{q-1}} + \frac{p-q}{p} \sum_{I \in \mathcal{S}} \frac{a_I y_I^p}{(\beta+1)^{q-1}} + y_X^{p-q} A, \end{aligned} \quad (3.9)$$

after we have expanded K . Note now that

$$\frac{1}{(\beta+1-\beta x)^{q-1}} - \frac{1}{(\beta+1)^{q-1}} \geq \frac{(q-1)\beta x}{(\beta+1)^q},$$

by the mean value theorem on derivatives for all $x \in [0, 1]$, so (3.9) becomes

$$\begin{aligned} F' &\geq \left(\frac{(q-1)\beta}{(\beta+1)^q} + \frac{p-q}{p(\beta+1)^{q-1}} \right) \int_X (\mathcal{M}_{\mathcal{T}}\phi)^p d\mu \\ &\quad - \frac{p-q}{p} \frac{f^p}{(\beta+1)^{q-1}} - f^{p-q} \frac{(q-1)\beta}{(\beta+1)^q} \int_X (\mathcal{M}_{\mathcal{T}}\phi)^q d\mu + f^{p-q} A \end{aligned} \quad (3.10)$$

for every $\beta > 0$. Rearranging the terms, we get (1.12) for \mathcal{T} -good functions.

For the general $\phi \in L^p(X, \mu)$ with $\int_X \phi = f$ and $\int_X \phi^q = A$, $f^q < A$, $1 < q < p$, (1.12) is proved as follows. We consider the sequence $\{\phi_m\}$, where $\phi_m = \sum_{I \in \mathcal{T}_{(m)}} \text{Av}_I(\phi) \chi_I$ and we set

$$\Phi_m = \sum_{I \in \mathcal{T}_{(m)}} \max\{\text{Av}_I(\phi) : I \subseteq J \in \mathcal{T}\} \chi_I = \mathcal{M}_{\mathcal{T}}\phi_m,$$

since $\text{Av}_J(\phi) = \text{Av}_J(\phi_m)$ whenever $I \subseteq J \in \mathcal{T}$. It is easy to see that

$$\int_X \phi_m d\mu = \int_X \phi d\mu = f, \quad \int_X \phi_m^q d\mu \leq \int_X \phi^q d\mu \quad (3.11)$$

for all m and that Φ_m converges monotonically almost everywhere to $\mathcal{M}_\mathcal{T}\phi$. Since ϕ_m is easily seen to be \mathcal{T} -good, from what we have just shown,

$$\begin{aligned} \int_X \Phi_m^p d\mu &\leq \frac{p(\beta+1)^q}{G(p,q,\beta)} \int_X \Phi_m^{p-q} \phi_m^q d\mu + \frac{(p-q)(\beta+1)}{G(p,q,\beta)} f^p \\ &\quad + \frac{p(q-1)\beta}{G(p,q,\beta)} f^{p-q} \int_X \Phi_m^q d\mu - \frac{p(\beta+1)^q}{G(p,q,\beta)} f^{p-q} \int_X \phi_m^q. \end{aligned} \quad (3.12)$$

Since \mathcal{T} differentiates $L^1(X, \mu)$ and by the definition of ϕ_m , if $\{I_m(x)\}$ is the chain of elements of \mathcal{T} which contain $x \in X$, then

$$\lim_{m \rightarrow \infty} \phi_m(x) = \lim_{m \rightarrow \infty} \text{Av}_{I_m(x)}(\phi) = \phi(x) \quad (3.13)$$

and $\phi_m \leq \Phi_m$. Taking limits using the monotone and dominated convergence theorems and Fatou's lemma, we obtain (1.12) for the general $\phi \in L^p(X, \mu)$.

4 Proof of Theorem 1

We now move on to show that (1.12) is sharp. To do this we shall use a result from [8] stated in Theorem 2 below. What makes it particularly useful in our case is that it is valid for any functions G_1, G_2 satisfying the specific properties mentioned. We remind here that the decreasing rearrangement $\phi^* : (0, \infty) \rightarrow [0, \infty]$ of a measurable function $\phi : X \rightarrow \mathbb{R}$, is defined as

$$\phi^*(t) = \inf\{s : d_\phi(s) \leq t\},$$

with d_ϕ the distribution function of ϕ .

Theorem 2. *The following is true*

$$\begin{aligned} &\sup \left\{ \int_K G_1(\mathcal{M}_\mathcal{T}\phi) G_2(\phi) d\mu : \phi^* = g, K \subseteq X \text{ measurable, with } \mu(K) = k \right\} \\ &= \int_0^k G_1\left(\frac{1}{t} \int_0^t g\right) G_2(g(t)) dt, \end{aligned} \quad (4.1)$$

where $G_i : [0, +\infty] \rightarrow [0, +\infty]$, $i = 1, 2$, are increasing functions, $g : (0, 1] \rightarrow \mathbb{R}$ is non-increasing and ϕ^* is the decreasing rearrangement of the function ϕ .

So, with X in place of K and from well known properties of the decreasing rearrangement, it is now easy to see that

$$\begin{aligned} &\sup \left\{ \int_X G_1(\mathcal{M}_\mathcal{T}\phi) G_2(\phi) d\mu : \phi \geq 0, \text{ measurable, with } \int_X \phi = f \right\} \\ &= \sup \left\{ \int_0^1 G_1\left(\frac{1}{t} \int_0^t g\right) G_2(g(t)) dt : g : (0, 1] \rightarrow \mathbb{R}, \text{ non-increasing, } \int_0^1 g = f \right\}. \end{aligned} \quad (4.2)$$

Let $\beta > 0$ and define $g : (0, 1] \rightarrow \mathbb{R}$ by

$$g(t) = \frac{f}{\beta + 1} t^{-1 + \frac{1}{\beta+1}}. \quad (4.3)$$

It is easy to see that $\int_0^1 g = f$, for every $\beta > 0$, $\frac{1}{t} \int_0^t g = (\beta + 1)g(t)$ for every $t \in (0, 1]$ and, after straightforward calculations, that

$$\begin{aligned} \int_0^1 \left(\frac{1}{t} \int_0^t g\right)^p d\mu &= \frac{p(\beta + 1)^q}{G(p, q, \beta)} \int_0^1 \left(\frac{1}{t} \int_0^t g\right)^{p-q} g^q d\mu + \frac{(p-q)(\beta + 1)}{G(p, q, \beta)} f^p \\ &+ \frac{p(q-1)\beta}{G(p, q, \beta)} f^{p-q} \int_0^1 \left(\frac{1}{t} \int_0^t g\right)^q d\mu - \frac{p(\beta + 1)^q}{G(p, q, \beta)} f^{p-q} \int_0^1 g^q d\mu. \end{aligned} \quad (4.4)$$

This, together with (4.2), proves the sharpness of (1.12) for the first case stated in Theorem 1. Now, (4.2) is valid if we add $\int_X \phi^q = A$ in the brackets of the left hand side of (4.2) and $\int_0^1 g^q = A$ to the right, that is if we consider the q -norms of the corresponding functions as fixed. So in case $\int_X \phi^q d\mu = A$, all we need to do is choose $\beta > 0$ so, that $\int_0^1 g^q = A$, with g as in (4.3). The appropriate value is easily seen to be the one given in Theorem 1 and the proof is complete.

5 Applications

Proof of Corollary 1.

Since $\int_X \phi d\mu = f$, from (4.25) in [3], we know that

$$\int_X (\mathcal{M}_T \phi)^q d\mu \leq \frac{\beta + 1}{\beta} \frac{(\beta + 1)^{q-1} \int_X \phi^q d\mu - f^q}{q - 1} \quad (5.1)$$

for every $\beta > 0$, for ϕ a \mathcal{T} -good function. Plugging this into (3.10) we get (1.13) for \mathcal{T} -good functions and defining ϕ_m and Φ_m as in Section 3, we get (1.13) for the general $\phi \in L^p(X, \mu)$, using the monotone convergence theorem. Sharpness is proved for both cases in the same way it has been proved for (1.12). We only need to observe that with g as in (4.3)

$$\int_0^1 \left(\frac{1}{t} \int_0^t g\right)^p d\mu = -\frac{q(\beta + 1)}{G(p, q, \beta)} f^p + \frac{p(\beta + 1)^q}{G(p, q, \beta)} \int_0^1 \left(\frac{1}{t} \int_0^t g\right)^{p-q} g^q d\mu. \quad (5.2)$$

□

Our final application is to derive the least upper bound for $\int_X (\mathcal{M}_T \phi)^p d\mu$, when on ϕ we impose the conditions $\int_X \phi d\mu = f$ and $\int_X \phi^p d\mu = F$ (where f, F are fixed, satisfying $0 \leq f^p \leq F$), by using the proof of inequality (1.13), for an arbitrary q belonging to $(1, p)$, and a suitable value of β , depending on q, p, f and F . That is we find the main Bellman function of two variables, (1.7), associated to the dyadic maximal operator. We proceed to this as follows.

Fix $q \in (1, p)$. First of all it is easy to see that for the above f, F , there exists $\beta \in (0, \frac{1}{p-1})$, such that

$$h_\beta(\beta+1)F = \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} f^p, \quad (5.3)$$

where $h_\beta(y)$ is defined, for every $y > 1$, by $h_\beta(y) = y^{p-q} - A_\beta y^p$ and A_β is defined by

$$A_\beta = \frac{(q-1)\beta}{(\beta+1)^q} + \frac{p-q}{p} \frac{1}{(\beta+1)^{q-1}}. \quad (5.4)$$

For this existence, we just need to define the function

$$G(\beta) = \frac{1}{(\beta+1)^{p-1}[1-\beta(p-1)]},$$

of $\beta \in (0, \frac{1}{p-1})$, and note that $G(0+) = 1$ and $G(\frac{1}{p-1}-) = +\infty$. Thus there exists $\beta \in (0, \frac{1}{p-1})$, such that $G(\beta) = \frac{F}{f^p} \geq 1$. If this last condition is true we easily see, after some simple calculations, that $h_\beta(\beta+1)F = \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} f^p$, which is (5.3).

Now, because of (1.13), for any $\phi \in L^p(X, \mu)$, and for this value of β , the following inequality holds

$$\int_X \phi^q (\mathcal{M}_T \phi)^{p-q} d\mu \geq A_\beta \int_X (\mathcal{M}_T \phi)^p d\mu + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} f^p.$$

Applying Hölder's inequality on the left side of the above inequality we obtain

$$F^{q/p} \left(\int_X (\mathcal{M}_T \phi)^p d\mu \right)^{(p-q)/p} \geq A_\beta \int_X (\mathcal{M}_T \phi)^p d\mu + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} f^p$$

or equivalently, by dividing both sides by F ,

$$I_\phi^{(p-q)/p} \geq A_\beta I_\phi + \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} \frac{f^p}{F},$$

where in the last inequality we denote $I_\phi = \frac{\int_X (\mathcal{M}_T \phi)^p d\mu}{F}$, which in turn means that

$$h_\beta(I_\phi^{1/p}) \geq \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} \frac{f^p}{F}. \quad (5.5)$$

Now for any $\beta \in (0, \frac{1}{p-1})$, we prove that the function h_β , with domain $(1, +\infty)$, is strictly decreasing. For this proof we proceed in the following way. We have that $\frac{d}{dy} h_\beta(y) = y^{p-1}[(p-q)y^{-q} - pA_\beta] < y^{p-1}[(p-q) - pA_\beta]$, where the inequality in the last relation is true due to the fact that y is greater than 1. Now we claim that $A_\beta > \frac{p-q}{p}$, for any $q \in [1, p]$ and $\beta \in (0, \frac{1}{p-1})$. For this reason, we consider A_β as a function of β , in the above mentioned domain and denote it as $K(\beta)$. Then $K(0) = \frac{p-q}{p}$, so we just need to prove that $K(\beta)$ is strictly increasing. For this purpose we evaluate $\frac{d}{d\beta} K(\beta)$, which as can be easily

seen by using (5.4) is equal to $\frac{(q-1)q[1-\beta(p-1)]}{p(\beta+1)^{q+1}}$, which is positive for any β as above. By the above discussion we conclude that $\frac{d}{dy}h_\beta(y) < 0$, for any $y > 1$.

Thus from (5.5) we have as a consequence that $I_\phi^{1/p} \leq h_\beta^{-1}(L)$, where $L = \frac{q}{p} \frac{1}{(\beta+1)^{q-1}} \frac{f^p}{F}$. This conclusion holds, if we suppose that $I_\phi > 1$, which may be assumed, since in the opposite case we have nothing to prove. We finally reach the inequality

$$\int_X (\mathcal{M}_T \phi)^p d\mu \leq F(h_\beta^{-1}(L))^p \quad (5.6)$$

Having now in mind that (5.3) holds, we show that $h_\beta^{-1}(L) = \omega_p\left(\frac{f^p}{F}\right)$, where ω_p is defined in the Introduction. Indeed, by (5.3), we immediately conclude that $h_\beta^{-1}(L) = \beta+1$, so we just need to prove that $\beta+1 = \omega_p\left(\frac{f^p}{F}\right)$. Equivalently this means that $H_p(\beta+1) = \frac{f^p}{F}$. But by (5.3), we easily see that

$$\frac{p}{q}(\beta+1)^{q-1}[(\beta+1)^{p-q} - A_\beta(\beta+1)^p] = \frac{f^p}{F}.$$

After simple calculations in the left side of the above equality, the real number q is cancelled giving us the quantity

$$-(p-1)(\beta+1)^p + p(\beta+1)^{p-1},$$

which is exactly $H_p(\beta+1)$. In this way we derive that

$$B_T^{(p)}(f, F) \leq F \omega_p\left(\frac{f^p}{F}\right)^p.$$

This establishes the least upper bound we need to find for the quantity of interest for the general $\phi \in L^p(X, \mu)$. Note finally that the opposite inequality is also true, as can be concluded immediately by the sharpness of inequality (1.13), which is best possible for any fixed values of f and β . Thus we have equality in the above inequality, and our evaluation of the Bellman function of two variables for the dyadic maximal operator is completed.

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